

MODIFIED ALGEBRAIC BETHE ANSATZ FOR XXZ CHAIN ON THE SEGMENT

- III - PROOF

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ABSTRACT. In this paper, we prove the off-shell equation satisfied by the transfer matrix associated with the XXZ spin- $\frac{1}{2}$ chain on the segment with two generic integrable boundaries acting on the Bethe vector. The essential step is to prove that the expression of the action of a modified creation operator on the Bethe vector has an off-shell structure which results in an inhomogeneous term in the eigenvalues and Bethe equations of the corresponding transfer matrix.

MSC: 82B23; 81R12

Keywords: algebraic Bethe ansatz; integrable spin chain; boundary conditions

1. INTRODUCTION

In the last decade, the study of the spectral problems associated with quantum integrable models which are not invariant by $U(1)$ symmetry has been an active research field. The prototype model is the XXZ spin- $\frac{1}{2}$ chain on the segment with general boundary conditions, see [4, 7] for the Hamiltonian in our notations. For non-diagonal boundaries, although the model is known to be integrable, the Bethe ansatz techniques, and in particular the algebraic Bethe ansatz (ABA) [29, 30], could not be directly applied due to the lack of a reference state for the transfer matrix¹. There were many attempts to overcome this problem: by introducing constraints between the left and the right boundary parameters [8, 32, 24, 25], by considering specific non-diagonal integrable boundaries² and introducing a new structure for the Bethe vector [6, 28], or by developing alternative methods to the Bethe ansatz [2, 17, 21], including the *separation of variables* (SoV) [26, 16] method. More historic details can be found in [4].

Recently, an essential step to construct the solution of quantum integrable models without $U(1)$ symmetry by means of the Bethe ansatz was accomplished by the introduction of the so-called *off-diagonal Bethe ansatz* (ODBA) [9, 10]. The key point is the use, as an ansatz to characterize the spectrum of such models, of a modified T-Q Baxter equation which exhibits an additional inhomogeneous term.

These results brought the questions: what are the Bethe vector associated with the new eigenvalues and, in addition, can the new inhomogeneous term be obtained solely from an ABA perspective?

The first progress arose in [5], where the Bethe vector of the XXX spin- $\frac{1}{2}$ chain on the segment were conjectured from the modified T-Q Baxter equation [10]. In [4, 7] the origin of the additional inhomogeneous term was clarified by means of a modified algebraic Bethe ansatz (MABA). In the MABA we drop the quest of the reference state and just focus on the construction of a highest weight vector for the underlying reflection algebra. In this framework, the additional term is understood as the action of a modified creation operator on the off-shell Bethe vector³. Such off-shell action was proved for the totally asymmetric simple exclusion process (TASEP) model (a degenerate case of the XXZ spin- $\frac{1}{2}$ chain on the segment with non-diagonal boundaries) in [12]. This proof uses the so-called F-basis [22] to analyze the pole structure of the creation

¹For models with $U(1)$ symmetry and q not root of unit, the reference state corresponds to the highest weight vector of the finite dimensional representation of the underlying algebra.

²In a way such that a reference state for the transfer matrix can still be identified.

³A Bethe vector is said to be off-shell if it depends on arbitrary parameters and on-shell if these parameters satisfy the Bethe equations.

operator. Let us also mention that a proof of the Bethe vector expression, solely for the on-shell case, was proposed from the ODBA for the XXX, in [33], and XXZ, in [34], spin- $\frac{1}{2}$ chains on the segment. In this case, the canonical SoV-basis [16] was used.

Here we prove the validity of MABA approach for the transfer matrix of the finite XXZ spin- $\frac{1}{2}$ chain on the segment with two generic boundaries. The expression of the off-shell Bethe vector of this model were conjectured in [7]. Let us recall that the construction of the Bethe vector follows the steps:

- Find the highest weight representation for the operators that describe the model through the generalized quantum inverse scattering method [30]. This was done in [8] from a local gauge transformation. This is possible if the dynamical right reflection matrix is diagonal or upper-triangular.
- Use the gauge transformation to map the transfer matrix to a modified lower triangular form [7] or to a modified diagonal form [5]. This step introduces the modified operator used to construct the Bethe vector (which is built from applying successively the modified operator on the highest weight vector).
- Apply the transfer matrix on the Bethe vector. Use the commutation relations and the action of the modified operators on the highest weight representation to obtain the off-shell action of the transfer matrix. This action has a new term with an additional modified creation operator acting on the Bethe vector.
- Show that the new term has a wanted-unwanted structure. From direct resolution of chains with small length ($N = 1, 2$), we notice that this is possible if the number of creation operators equals the length of the chain. We observe that these wanted-unwanted terms give the inhomogeneous part in the eigenvalues and in the Bethe equations of the transfer matrix introduced in the context of the ODBA. This step was performed in [7].

In this paper, we prove the explicit form of the off-shell action of the modified creation operator on the off-shell Bethe vector using the canonical SoV-basis [16]. As an additional result, we make the explicit link between the MABA and the SoV [16, 26] and ODBA [10, 33, 34] methods.

The paper is organized as follows. In section 2, we recall the definition of the transfer matrix of XXZ spin- $\frac{1}{2}$ chain on the segment in the generalized quantum inverse scattering framework as well as the gauge transformation that yields the dynamical operators. In section 3, we recall the highest weight representation of the dynamical operators and introduce the modified dynamical operators. In section 4, we implement the MABA and in section 5 we prove the action formula of the modified creation operator on the off-shell Bethe vector. The connection with the ODBA and SoV methods is given in section 6. We conclude in section 7.

2. GENERALIZED QUANTUM INVERSE SCATTERING AND GAUGE TRANFORMATION

The transfer matrix of the finite XXZ spin- $\frac{1}{2}$ chain on the segment is constructed following the generalized quantum inverse scattering framework [30]. Here we recall only the essential features, more details in our notations can be found in [4, 7]. From the trigonometric R -matrix, which acts on $V_a \otimes V_b$ ⁴,

$$(2.1) \quad R_{ab}(u) = \begin{pmatrix} b(qu) & 0 & 0 & 0 \\ 0 & b(u) & 1 & 0 \\ 0 & 1 & b(u) & 0 \\ 0 & 0 & 0 & b(qu) \end{pmatrix}, \quad b(u) = \frac{u - u^{-1}}{q - q^{-1}},$$

⁴In this work, we only use two-dimensional complex vector spaces, *i.e.* $V = \mathbb{C}^2$.

and from the K -matrices [15], which act on V ,

$$(2.2) \quad K^-(u) = \begin{pmatrix} k^-(u) & \tau^2 c(u) \\ \tilde{\tau}^2 c(u) & k^-(u^{-1}) \end{pmatrix}, \quad k^-(u) = \nu_- u + \nu_+ u^{-1}, \quad c(u) = u^2 - u^{-2},$$

$$(2.3) \quad K^+(u) = \begin{pmatrix} k^+(qu) & \tilde{\kappa}^2 c(qu) \\ \kappa^2 c(qu) & k^+(q^{-1}u^{-1}) \end{pmatrix}, \quad k^+(u) = \epsilon_+ u + \epsilon_- u^{-1},$$

where $\{\epsilon_{\pm}, \kappa, \tilde{\kappa}\}$ and $\{\nu_{\pm}, \tau, \tilde{\tau}\}$ are generic parameters, one constructs the transfer matrix, that acts on the Hilbert space $\mathcal{H} = \otimes_{i=1}^N V_i$,

$$(2.4) \quad t(u) = \text{tr}_a(K_a^+(u)K_a(u)),$$

where

$$(2.5) \quad K_a(u) = R_{a1}(u/v_1) \dots R_{aN}(u/v_N) K_a^-(u) R_{aN}(uv_N) \dots R_{a1}(uv_1),$$

$$(2.6) \quad = \begin{pmatrix} \mathcal{A}(u) & \mathcal{B}(u) \\ \mathcal{C}(u) & \mathcal{D}(u) + \frac{1}{b(qu^2)} \mathcal{A}(u) \end{pmatrix}_a,$$

is the double row monodromy matrix that acts on $V_a \otimes \mathcal{H}$. This formalism allows one to reformulate the problem in the operator language. The transfer matrix is given by,

$$(2.7) \quad t(u) = \phi(u) k^+(u) \mathcal{A}(u) + k^+(q^{-1}u^{-1}) \mathcal{D}(u) + c(qu) \left(\kappa^2 \mathcal{B}(u) + \tilde{\kappa}^2 \mathcal{C}(u) \right),$$

with

$$(2.8) \quad \phi(u) = \frac{b(q^2 u^2)}{b(qu^2)}.$$

An important point in the ABA framework is the existence of a highest weight vector which is annihilated by the operator $\mathcal{C}(u)$ and which is an eigenstate of the operators $\mathcal{A}(u)$ and $\mathcal{D}(u)$. This vector exists when we have an upper triangular K^- matrix. However, for generic K^- matrix, one needs to use the gauge transformation [1, 8] in order to define new double-row operators that are dynamical, *i.e.* they depend of an additional parameter said dynamical, and to be able to construct such highest weight vector. In addition, the gauge transformation allow us to eliminate the dynamical creation and/or annihilation operators in the transfer matrix expression. The building blocks of the gauge transformation are the covariant vectors,

$$(2.9) \quad |X(u, m)\rangle = \begin{pmatrix} \alpha q^{-m} u^{-1} \\ 1 \end{pmatrix}, \quad |Y(u, m)\rangle = \begin{pmatrix} \beta q^m u^{-1} \\ 1 \end{pmatrix},$$

and the contravariant vectors,

$$(2.10) \quad |\tilde{X}(u, m)\rangle = \frac{qu}{\gamma_{m-1}} \begin{pmatrix} -1, & \alpha q^{-m} u^{-1} \end{pmatrix}, \quad \langle \tilde{Y}(u, m)| = \frac{qu}{\gamma_{m+1}} \begin{pmatrix} 1, & -\beta q^m u^{-1} \end{pmatrix},$$

where α and β are generic complex parameters, m is an integer and $\gamma(u, m) = \alpha q^{-m} u - \beta q^m u^{-1}$ with $\gamma(1, m) = \gamma_m$. They satisfy the shifted scalar products,

$$(2.11) \quad \langle \tilde{X}(u, m)|X(u, m)\rangle = \langle \tilde{Y}(u, m)|Y(u, m)\rangle = 0,$$

$$(2.12) \quad \langle \tilde{X}(u, m+1)|Y(u, m-1)\rangle = \langle \tilde{Y}(u, m-1)|X(u, m+1)\rangle = 1,$$

as well as the shifted closure relation,

$$(2.13) \quad |Y(u, m-1)\rangle \langle \tilde{X}(u, m+1)| + |X(u, m+1)\rangle \langle \tilde{Y}(u, m-1)| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

From these vectors we can introduce the dynamical operators,

$$(2.14) \quad \mathcal{C}(u, m) = \langle \tilde{X}(u, m) | K_a(u) | X(u^{-1}, m) \rangle,$$

$$(2.15) \quad \mathcal{B}(u, m) = \langle \tilde{Y}(u, m) | K_a(u) | Y(u^{-1}, m) \rangle,$$

$$(2.16) \quad \mathcal{A}(u, m) = \langle \tilde{Y}(u, m-2) | K_a(u) | X(u^{-1}, m) \rangle,$$

$$(2.17) \quad \mathcal{D}(u, m) = \frac{\gamma_{m+1}}{\gamma_m} \langle \tilde{X}(u, m+2) | K_a(u) | Y(u^{-1}, m) \rangle - \frac{\gamma(u^{-2}, m+1)}{b(qu^2)\gamma_m} \mathcal{A}(u, m).$$

such that the transfer matrix can be written in the following way,

$$(2.18) \quad t(u) = t_d(u, m) + u^{-1} c(qu) \left(\zeta_m \mathcal{B}(u, m) - \tilde{\zeta}_m \mathcal{C}(u, m) \right),$$

with

$$(2.19) \quad \zeta_m = \frac{\kappa^2}{\gamma_m} \left(\alpha q^{-m-1} + i \frac{\tilde{\kappa}\xi}{\kappa\tilde{\xi}} \right) \left(\alpha q^{-m-1} + i \frac{\tilde{\kappa}\tilde{\xi}}{\kappa\xi} \right), \quad \tilde{\zeta}_m = \frac{\kappa^2}{\gamma_m} \left(\beta q^{m-1} + i \frac{\tilde{\kappa}\xi}{\kappa\tilde{\xi}} \right) \left(\beta q^{m-1} + i \frac{\tilde{\kappa}\tilde{\xi}}{\kappa\xi} \right),$$

and the new parametrization,

$$(2.20) \quad \epsilon_- = i\tilde{\kappa}\kappa(\xi/\tilde{\xi} + \tilde{\xi}/\xi), \quad \epsilon_+ = i\tilde{\kappa}\kappa(\xi\tilde{\xi} + 1/(\tilde{\xi}\xi)).$$

The term $t_d(u, m)$, which is the diagonal part of the transfer matrix, is given by

$$(2.21) \quad t_d(u, m) = \tilde{a}(u, m) \mathcal{A}(u, m) + \tilde{d}(u, m) \mathcal{D}(u, m),$$

where

$$(2.22) \quad \tilde{a}(u, m) = \tilde{a}(u) - \delta_m \phi(q^{-1}u^{-1}) u^{-1} c(qu), \quad \tilde{d}(u, m) = \tilde{d}(u) + u^{-1} c(qu) \delta_m$$

with

$$(2.23) \quad \tilde{a}(u) = u^{-1} \phi(u) \tilde{k}^+(u), \quad \tilde{d}(u) = u^{-1} \tilde{k}^+(q^{-1}u^{-1}),$$

$$(2.24) \quad \tilde{k}^+(u) = i\tilde{\kappa}\kappa(\tilde{\xi}u + \tilde{\xi}^{-1}u^{-1})(\xi^{-1}u + \xi u^{-1}),$$

and

$$(2.25) \quad \delta_m = \frac{\kappa^2}{\gamma_{m+1}} \left(\alpha q^{-m-1} + i \frac{\tilde{\kappa}\xi}{\kappa\tilde{\xi}} \right) \left(\beta q^{m+1} + i \frac{\tilde{\kappa}\tilde{\xi}}{\kappa\xi} \right).$$

Let us introduce finally a bookkeeping notation. For generic gauge parameters $\{\alpha, \beta\}$ the dynamical operators are denoted by,

$$\{\mathcal{A}(u, m), \mathcal{B}(u, m), \mathcal{C}(u, m), \mathcal{D}(u, m)\}.$$

If we consider some fixed gauge parameters $\{\alpha_x, \beta_x\}$ we denote the corresponding fixed dynamical operators with indices, namely,

$$\{\mathcal{A}_x(u, m), \mathcal{B}_x(u, m), \mathcal{C}_x(u, m), \mathcal{D}_x(u, m)\}.$$

3. REPRESENTATION THEORY AND MODIFIED OPERATORS

In order to consider the diagonalization of the transfer matrix in the framework of the generalized quantum inverse scattering method, we want first to introduce a suitable basis to construct the Bethe vector. In particular, since we want a highest weight representation, it can be built for the dynamical operators with the choice of gauge parameters

$$(3.1) \quad \alpha \Rightarrow \alpha_{dr} = iq^{m_0+N} \frac{\tau\mu}{\tilde{\tau}\tilde{\mu}}, \quad \beta \Rightarrow \beta_{dr} = iq^{-m_0-N} \frac{\tau\tilde{\mu}}{\tilde{\tau}\mu},$$

which corresponds to bring the dynamical right K^- -matrix to a diagonal form, see [8] for details. As a consequence, it allows us to construct the highest weight vector,

$$(3.2) \quad |\Omega_{m_0}^N\rangle_{dr} = \otimes_{i=1}^N |X(v_i, m_0 + i)\rangle_{dr}.$$

such that the actions of the dynamical operators at the point $m = m_0$ are given by,

$$(3.3) \quad \mathcal{A}_{dr}(u, m_0) |\Omega_{m_0}^N\rangle_{dr} = u \tilde{k}^-(u) \Lambda(u) |\Omega_{m_0}^N\rangle_{dr},$$

$$(3.4) \quad \mathcal{D}_{dr}(u, m_0) |\Omega_{m_0}^N\rangle_{dr} = u \phi(q^{-1}u^{-1}) \tilde{k}^-(q^{-1}u^{-1}) \Lambda(q^{-1}u^{-1}) |\Omega_{m_0}^N\rangle_{dr},$$

$$(3.5) \quad \mathcal{C}_{dr}(u, m_0) |\Omega_{m_0}^N\rangle_{dr} = 0,$$

where

$$(3.6) \quad \Lambda(u) = \prod_{i=1}^N b(qu/v_i) b(quv_i),$$

and

$$(3.7) \quad \tilde{k}^-(u) = i\tilde{\tau}\tau(\mu u + \mu^{-1}u^{-1})(\tilde{\mu}^{-1}u + \tilde{\mu}u^{-1}),$$

with the new parametrization

$$(3.8) \quad \nu_- = i\tilde{\tau}\tau(\mu/\tilde{\mu} + \tilde{\mu}/\mu), \quad \nu_+ = i\tilde{\tau}\tau(\mu\tilde{\mu} + 1/(\mu\tilde{\mu})).$$

Moreover the dynamical operator \mathcal{B}_{dr} is dynamically nilpotent of order $N + 1$,

$$(3.9) \quad \mathcal{B}_{dr}(u, m_0 + 2N) \mathcal{B}_{dr}(u_1, m_0 + 2(N-1)) \dots \mathcal{B}_{dr}(u_N, m_0) = 0,$$

where $\bar{u} = \{u_1, \dots, u_N\}$ is a set of N generic parameters.

Let us now consider the string of M operators acting on the highest weight vector,

$$(3.10) \quad \mathcal{B}_{dr}(u_{j_1}, m_0 + 2(M-1)) \dots \mathcal{B}_{dr}(u_{j_M}, m_0) |\Omega_{m_0}^N\rangle_{dr}$$

where $M \in \{0, \dots, N\}$ and $\{u_{j_1}, \dots, u_{j_M}\}$ is some subset of cardinality M of \bar{u} . Since the vectors (3.10) are independent for generic \bar{u} , from these vectors and for fixed M we generate a subspace $\mathcal{W}_M^{m_0}$ of dimension $\binom{N}{M} = \frac{N!}{M!(N-M)!}$ in the Hilbert space. The highest weight representation induces the decomposition

$$(3.11) \quad \mathcal{H} = \bigoplus_{M=0}^N \mathcal{W}_M^{m_0}$$

of the Hilbert space. Indeed the total dimension of the sum on the r.h.s. is

$$(3.12) \quad \sum_{M=0}^N \binom{N}{M} = 2^N.$$

Thus, in order to construct the Bethe vector of the model, we are looking for a linear combination of the vectors (3.10). However, fixing the full family of coefficients is a hard task. It was performed for small N in the case of the XXX chain on the segment [5], by using the *a priori* knowledge of the eigenvalues. It appears that the Bethe vector factorizes in terms of a modified creation operator. This modified creation operator and the associated family of other modified operators are the key step to implement the MABA without the knowledge of the eigenvalues. These operators are linear combinations of the ones used to construct the highest weight representation. This modified family of operators reduces the transfer matrix to a modified diagonal form (see below) and the commutation relations are similar to the diagonal boundary case. Thus, a part of the calculation of the MABA is similar to the usual ABA and reduces to already known cases by simple limit. Indeed, for the choice of the gauge parameters,

$$(3.13) \quad \alpha \Rightarrow \alpha_{dl}(M) = -iq^{1+m_0+2M} \frac{\xi \tilde{\kappa}}{\xi \kappa}, \quad \beta \Rightarrow \beta_{dl}(M) = -iq^{1-m_0-2M} \frac{\tilde{\xi} \tilde{\kappa}}{\xi \kappa},$$

the dynamical transfer matrix (2.21) at the point $m_0 + 2M$ takes a modified diagonal form, *i.e.* it depends only in \mathcal{A} and \mathcal{D} operators, namely

$$(3.14) \quad t(u) = t_{dl}(u, m_0 + 2M) = \tilde{a}(u) \mathcal{A}_{dl}(u, m_0 + 2M) + \tilde{d}(u) \mathcal{D}_{dl}(u, m_0 + 2M).$$

Thus it becomes natural to consider $\mathcal{B}_{dl}(u, m)$ as a modified creation operator that we will use to construct the Bethe vector. In the following, to apply the ABA, we will need the commutation relations

$$(3.15) \quad \mathcal{B}_{dl}(u, m+2) \mathcal{B}_{dl}(v, m) = \mathcal{B}_{dl}(v, m+2) \mathcal{B}_{dl}(u, m),$$

$$\mathcal{A}_{dl}(u, m+2) \mathcal{B}_{dl}(v, m) = f(u, v) \mathcal{B}_{dl}(v, m) \mathcal{A}_{dl}(u, m) +$$

$$(3.16) \quad g_{dl}(u, v, m) \mathcal{B}_{dl}(u, m) \mathcal{A}_{dl}(v, m) + w_{dl}(u, v, m) \mathcal{B}_{dl}(u, m) \mathcal{D}_{dl}(v, m),$$

$$\mathcal{D}_{dl}(u, m+2) \mathcal{B}_{dl}(v, m) = h(u, v) \mathcal{B}_{dl}(v, m) \mathcal{D}_{dl}(u, m)$$

$$(3.17) \quad + k_{dl}(u, v, m) \mathcal{B}_{dl}(u, m) \mathcal{D}_{dl}(v, m) + n_{dl}(u, v, m) \mathcal{B}_{dl}(u, m) \mathcal{A}_{dl}(v, m),$$

obtained from the method described in [8] and with functions defined in the appendix A. See also [7] for details.

An important fact is that the operators of the two families are related by linear relations. In particular we have

$$(3.18) \quad \mathcal{A}_{dl}(u, p) = \mathcal{A}_{dr}(u, m) + c(p, m) \mathcal{C}_{dr}(u, m) + b(p, m) \mathcal{B}_{dl}(u, p-2)$$

$$(3.19) \quad \mathcal{D}_{dl}(u, p) = \mathcal{D}_{dr}(u, m) - \phi(u) c(p, m) \mathcal{C}_{dr}(u, m) - \phi(u) b(p, m) \mathcal{B}_{dl}(u, p-2)$$

with

$$(3.20) \quad b(p, m) = \frac{\alpha_{dr} - q^{m-p} \alpha_{dl}}{\alpha_{dr} - q^{m+p-2} \beta_{dl}}, \quad c(p, m) = \frac{q^{m-p} \beta_{dr} - \beta_{dl}}{q^{2-m-p} \alpha_{dr} - \beta_{dl}}.$$

It allows to find the action of the operators of the modified transfer matrix (2.21) on the highest weight vector (3.2), namely

$$(3.21) \quad \mathcal{A}_{dl}(u, m_0) |\Omega_{m_0}^N\rangle_{dr} = u \tilde{k}^-(u) \Lambda(u) |\Omega_{m_0}^N\rangle_{dr} + \eta_{m_0} \mathcal{B}_{dl}(u, m_0-2) |\Omega_{m_0}^N\rangle_{dr},$$

$$(3.22) \quad \mathcal{D}_{dl}(u, m_0) |\Omega_{m_0}^N\rangle_{dr} = u \phi(q^{-1}u^{-1}) \tilde{k}^-(q^{-1}u^{-1}) \Lambda(q^{-1}u^{-1}) |\Omega_{m_0}^N\rangle_{dr} - \phi(u) \eta_{m_0} \mathcal{B}_{dl}(u, m_0-2) |\Omega_{m_0}^N\rangle_{dr},$$

with

$$(3.23) \quad \eta_{m_0} = b(m_0, m_0) = \frac{\alpha_{dr} - \alpha_{dl}}{\alpha_{dr} - \beta_{dl} q^{2m_0-2}}.$$

Remark 3.1. These actions have an off-diagonal contribution characteristic of the MABA already pointed out in [5, 4, 7].

4. MABA

From the modified dynamical \mathcal{B}_{dl} operator, used as creation operator, we construct the string of M operators

$$(4.1) \quad B_{dl}(\bar{u}, m, M) = \mathcal{B}_{dl}(u_1, m+2(M-1)) \dots \mathcal{B}_{dl}(u_M, m),$$

and act with it on the highest weight vector (3.2). It yields the vector,

$$(4.2) \quad |\Psi_{m_0}^M(\bar{u})\rangle = B_{dl}(\bar{u}, m_0, M) |\Omega_{m_0}^N\rangle_{dr}.$$

Using the commutation relations (3.15,3.16,3.17), the off-diagonal actions on the highest weight vector (3.21,3.22), and the functional relations (A.8,A.9) to simplify the term with the additional \mathcal{B}_{dl} operator, we obtain from the usual algebraic Bethe ansatz arguments, see [7] for more details, the off-shell action for the modified transfer matrix,

$$(4.3) \quad t_{dl}(u, m_0 + 2M) |\Psi_{m_0}^M(\bar{u})\rangle = \Lambda_d^M(u, \bar{u}) |\Psi_{m_0}^M(\bar{u})\rangle + \sum_{i=1}^M \tilde{F}(u, u_i) E_d^M(u_i, \bar{u}_i) |\Psi_{m_0}^M(\{u, \bar{u}_i\})\rangle \\ + \hat{\eta}_{m_0} u^{-1} c(qu) B_{dl}(\{u, \bar{u}\}, m_0 - 2, M + 1) |\Omega_{m_0}^N\rangle_{dr},$$

with

$$(4.4) \quad \Lambda_d^M(u, \bar{u}) = \phi(u) \tilde{k}^+(u) \tilde{k}^-(u) \Lambda(u) f(u, \bar{u}) + \phi(q^{-1}u^{-1}) \tilde{k}^+(q^{-1}u^{-1}) \tilde{k}^-(q^{-1}u^{-1}) \Lambda(q^{-1}u^{-1}) h(u, \bar{u}),$$

$$(4.5) \quad E_d^M(u_i, \bar{u}_i) = \phi(q^{-1}u_i^{-1}) \phi(u_i) \left(\tilde{k}^+(u_i) \tilde{k}^-(u_i) \Lambda(u_i) f(u_i, \bar{u}_i) \right. \\ \left. - \tilde{k}^+(q^{-1}u_i^{-1}) \tilde{k}^-(q^{-1}u_i^{-1}) \Lambda(q^{-1}u_i^{-1}) h(u_i, \bar{u}_i) \right),$$

$$(4.6) \quad \hat{\eta}_{m_0} = q^{-1} \kappa^2 \gamma_{m_0-1}^{dl} \eta_{m_0}.$$

and where we denote the set $\bar{u} = \{u_1, \dots, u_M\}$ with cardinality M , the set $\bar{u}_i = \{u_1, u_2, \dots, u_{i-1}, u_{i+1}, \dots, u_M\}$, where the element u_i is removed, and the set $\{u, \bar{u}_i\}$ as the union of u and \bar{u}_i .

Remark 4.1. *From the requirement $\hat{\eta}_{m_0} = 0$ we recover the result of [8] which implies constraints between left and right boundary parameters. See [4, 7] for more details about the limit cases.*

The new feature, characteristic of the MABA, is the presence of the string of $M + 1$ creation operators in the off-shell equation (4.3). This term can be handled according to the following proposition,

Proposition 4.1. *For $M = N$, the length of the spin chain, the string of $N + 1$ modified creation operators \mathcal{B}_{dl} acting on the highest weight vector (3.2) has an off-shell structure given by,*

$$(4.7) \quad \hat{\eta}_{m_0} u^{-1} c(qu) B_{dl}(\{u, \bar{u}\}, m_0 - 2, N + 1) |\Omega_{m_0}^N\rangle_{dr} = \Lambda_g^N(u, \bar{u}) |\Psi_{m_0}^N(\bar{u})\rangle \\ + \sum_{i=1}^N \tilde{F}(u, u_i) E_g^N(u_i, \bar{u}_i) |\Psi_{m_0}^N(\{u, \bar{u}_i\})\rangle$$

with

$$(4.8) \quad \Lambda_g^N(u, \bar{u}) = \chi c(u) c(q^{-1}u^{-1}) \Lambda(u) \Lambda(q^{-1}u^{-1}) G(u, \bar{u}),$$

$$(4.9) \quad E_g^N(u_i, \bar{u}_i) = -\chi \frac{c(u_i) c(q^{-1}u_i^{-1})}{b(qu_i^2)} \Lambda(u_i) \Lambda(q^{-1}u_i^{-1}) G(u_i, \bar{u}_i).$$

and

$$(4.10) \quad \chi = -\kappa \tilde{\kappa} \tau \tilde{\tau} \left(\frac{\kappa \tau}{\tilde{\kappa} \tilde{\tau}} + \frac{\tilde{\kappa} \tilde{\tau}}{\kappa \tau} + \frac{\xi \tilde{\mu}}{\tilde{\xi} \mu} q^{N+1} + \frac{\tilde{\xi} \mu}{\xi \tilde{\mu}} q^{-N-1} \right).$$

The function $\Lambda(u)$ is given by (3.6) and the constant factor $\hat{\eta}_{m_0}$ by (4.6). The auxiliary functions $c(u)$, $b(u)$, $\tilde{F}(u, v)$ and $G(u, v)$ are collected in appendix A, see equations (A.1) and (A.2).

The proof of this proposition will be postponed to the next section.

Remark 4.2. *This new term gives the inhomogeneous part of the modified T-Q Baxter equation introduced in the ODBA approach [10] and recovered from the SoV approach in [20].*

Remark 4.3. *The expression of the off-shell action of the modified creation operators \mathcal{B}_{dl} conjectured in [7] have additional terms. These terms come from the shift of m_0 by -2 of the l.h.s of the equation (4.7) here by comparison of the l.h.s of the equation (5.13) in [7]. This difference is due to the way we apply the MABA: in [7], we put the transfer matrix into a triangular modified form and here we put the transfer matrix into a diagonal modified form. In both cases the off-shell Bethe vector are the same, up to an overall constant.*

The proposition 4.1 allows us to write the main result of this paper, *i.e.*, the expression of the off-shell action of the transfer matrix with two general boundaries on the Bethe vector (4.2),

$$(4.11) \quad t(u)|\Psi_{m_0}^N(\bar{u})\rangle = \Lambda^N(u, \bar{u})|\Psi_{m_0}^N(\bar{u})\rangle + \sum_{i=1}^N \tilde{F}(u, u_i)E^N(u_i, \bar{u}_i)|\Psi_{m_0}^N(\{u, \bar{u}_i\})\rangle$$

with

$$(4.12) \quad \Lambda^N(u, \bar{u}) = \Lambda_d^N(u, \bar{u}) + \Lambda_g^N(u, \bar{u}), \quad E^N(u_i, \bar{u}_i) = E_d^N(u_i, \bar{u}_i) + E_g^N(u_i, \bar{u}_i),$$

where the modified diagonal part is given by (4.4,4.5) and the off-diagonal contribution by (4.8,4.9). For $E^N(u_i, \bar{u}_i) = 0$, which are the Bethe's equations of the model, we obtain the on-shell action of the transfer matrix which describes the eigenproblem of the transfer matrix (2.18).

5. PROOF OF THE OFF-SHELL ACTION (4.7)

In this section we prove the off-shell equation (4.7), which is the core equation of the MABA. We will use the left SoV basis of the dynamical \mathcal{B}_{dl} operator constructed in [16]. Let us consider the left-vector given by the product of the contravariant vectors (2.10),

$$(5.1) \quad {}_{dl}\langle \Omega_m^N | = \otimes_{n=1}^N {}_{dl}\langle \tilde{Y}(v_n, m+n) |$$

It is shown in [16] that it is a pseudo-eigenvector of the dynamical \mathcal{B}_{dl} operator⁵, namely,

$$(5.2) \quad {}_{dl}\langle \Omega_m^N | \mathcal{B}_{dl}(u, m) = {}_{dl}\langle \Omega_{m-2}^N | \eta_m^N \Lambda_b(u)$$

with

$$(5.3) \quad \eta_m^N = -q\tilde{\tau}^2 \left(q^{N+m} \beta_{dl} - i \frac{\tilde{\mu}\tau}{\mu\tilde{\tau}} \right) \left(q^{N+m} \beta_{dl} - i \frac{\mu\tau}{\tilde{\mu}\tilde{\tau}} \right) \frac{\gamma_m^{dl}}{\gamma_{m+N}^{dl} \gamma_{m+N+1}^{dl}}$$

and

$$(5.4) \quad \Lambda_b(u) = u c(u) \prod_{n=1}^N b(qu/v_n) b(uv_n).$$

It follows that the vectors

$$(5.5) \quad {}_{dl}\langle \tilde{\Psi}_m(\bar{h}) | = {}_{dl}\langle \Omega_m^N | \mathcal{A}_{dl}^{h_1}(v_1^{-1}, m+2) \dots \mathcal{A}_{dl}^{h_N}(v_N^{-1}, m+2)$$

with $\bar{h} = \{h_1, \dots, h_N\}$ and $h_i \in \{0, 1\}$, $i = 1, \dots, N$, are pseudo-eigenvectors of the dynamical \mathcal{B}_{dl} operator,

$$(5.6) \quad {}_{dl}\langle \tilde{\Psi}_m(\bar{h}) | \mathcal{B}_{dl}(u, m) = \left(\eta_m^N \Lambda_b(u) \prod_{i=1}^N f(v_i^{-1}, u)^{h_i} \right) {}_{dl}\langle \tilde{\Psi}_{m-2}(\bar{h}) |$$

and its pseudo-spectrum is simple for generic inhomogeneity parameters, see [16] for details. Thus the SoV basis (5.5) describe the full Hilbert space \mathcal{H} .

Then, using the left-SoV-basis (5.5), we can project the equation (4.7) on the 2^N one-dimensional subsets of the Hilbert space \mathcal{H} . It reduces the equation (4.7) to functional relations which we can prove by usual

⁵The term pseudo-eigenvector rather than eigenvector is due to the shift of m .

analytical methods. In fact, let us first consider the projection of the string of dynamical operators \mathcal{B}_{dl} . From the properties of the SoV-basis we find,

$$(5.7) \quad dl\langle \tilde{\Psi}_{m_0+2(N-1)}(\bar{h}) | B_{dl}(\{\bar{u}, u\}, m_0 - 2, N + 1) | \Omega_{m_0}^N \rangle_{dr} = \\ \Lambda_b(u) \Lambda_b(\bar{u}) W(\bar{h}) \prod_{i=1}^{N+1} \eta_{m_0+2(N-i)}^N \left(\prod_{i=1}^N f(v_i^{-1}, u)^{h_i} f(v_i^{-1}, \bar{u})^{h_i} \right) dl\langle \Omega_{m_0-4}^N | \Omega_{m_0}^N \rangle_{dr},$$

$$(5.8) \quad dl\langle \tilde{\Psi}_{m_0+2(N-1)}(\bar{h}) | B_{dl}(\{\bar{u}_j, u\}, m_0, N) | \Omega_{m_0}^N \rangle_{dr} = \\ \Lambda_b(u) \Lambda_b(\bar{u}_j) W(\bar{h}) \prod_{i=1}^N \eta_{m_0+2(N-i)}^N \left(\prod_{i=1}^N f(v_i^{-1}, u)^{h_i} f(v_i^{-1}, \bar{u}_j)^{h_i} \right) dl\langle \Omega_{m_0-2}^N | \Omega_{m_0}^N \rangle_{dr},$$

where we have used the formula,

$$(5.9) \quad dl\langle \tilde{\Psi}_m(\bar{h}) | \Omega_{m_0}^N \rangle_{dr} = W(\bar{h}) dl\langle \Omega_m^N | \Omega_{m_0}^N \rangle_{dr},$$

with

$$(5.10) \quad W(\bar{h}) = \prod_{i=1}^N \left(v_i^{-1} \tilde{k}^-(v_i^{-1}) \Lambda(v_i^{-1}) \right)^{h_i}$$

which follows from the recursion relation⁶,

$$(5.11) \quad dl\langle \tilde{\Psi}_m(\bar{h}) | \Omega_{m_0}^N \rangle_{dr} = \left(v_N^{-1} \tilde{k}^-(v_N^{-1}) \Lambda(v_N^{-1}) \right)^{h_N} dl\langle \tilde{\Psi}_{m-2}(\bar{h}_N) | \Omega_{m_0}^N \rangle_{dr}$$

and where we have

$$(5.12) \quad dl\langle \Omega_m^N | \Omega_{m_0}^N \rangle_{dr} = \prod_{n=1}^N dl\langle \tilde{Y}(v_n, m+n) | X(v_n, m_0+n) \rangle_{dr} = \prod_{n=1}^N \frac{q(\alpha_{dr} q^{-m_0-n} - \beta_{dl} q^{m+n})}{\gamma_{m+n+1}}.$$

By means of these projections, we reduce the matrix equation (4.7) to the functional equations,

$$(5.13) \quad -\chi u^{-1} c(qu) = \frac{\Lambda_g^N(u, \bar{u})}{\Lambda_b(u) \prod_{j=1}^N f(v_j^{-1}, u)^{h_j}} + \sum_{i=1}^N \tilde{F}(u, u_i) \frac{E_g^N(u_i, \bar{u}_i)}{\Lambda_b(u_i) \prod_{j=1}^N f(v_j^{-1}, u_i)^{h_j}},$$

where we use

$$(5.14) \quad \hat{\eta}_{m_0} \eta_{m_0-2}^N \frac{dl\langle \Omega_{m_0-4}^N | \Omega_{m_0}^N \rangle_{dr}}{dl\langle \Omega_{m_0-2}^N | \Omega_{m_0}^N \rangle_{dr}} = -\chi.$$

Using the explicit form (4.8,4.9), we reduce the proof to the checking the 2^N functional relations,

$$(5.15) \quad \sum_{i=1}^N \frac{1}{b(quu_i)b(u/u_i)} \frac{P(u_i, \bar{v}, \bar{h})}{\prod_{n \neq i} b(qu_iu_n)b(u_i/u_n)} = \frac{P(u, \bar{v}, \bar{h})}{\prod_{n=1}^N b(quu_n)b(u/u_n)} - 1$$

with

$$(5.16) \quad P(u, \bar{v}, \bar{h}) = \prod_{i=1}^N \left(b(uv_i) b(qu/v_i) \right)^{h_i} \left(b(quv_i) b(u/v_i) \right)^{1-h_i}.$$

It can be proved by considering left and right side of the equation (5.15) as two meromorphic functions of u and showing that they have the same poles and residues (at $u = u_i$ and $u = q^{-1}u_i^{-1}$) on the full complex plane and the same values at $u = \infty$ and $u = 0$. One uses the property $P(u, \bar{v}, \bar{h}) = P(q^{-1}u^{-1}, \bar{v}, \bar{h})$.

⁶This recursion relation is proved in [34].

6. CONNECTION WITH THE ODBA AND THE SoV METHODS

The ODBA and the SoV methods use the analytic properties of the transfer matrix. It is a Laurent series in u of degree $2N + 4$ and satisfy the parity condition and the crossing symmetry, see *e. g.* [10],

$$(6.1) \quad t(-u) = t(u), \quad t(q^{-1}u^{-1}) = t(u).$$

It follows from these two properties that the transfer matrix can be rewritten as a polynomial of

$$(6.2) \quad U(u) = \frac{qu^2 + q^{-1}u^{-2}}{q + q^{-1}}$$

of degree $N + 2$ with coefficients t_i acting on \mathcal{H} .

$$(6.3) \quad t(u) = \sum_{i=0}^{N+2} t_i U(u)^i.$$

We note that this variable $U(u)$ is used to extract the q-Onsager algebra from the reflection equation [2, 3]. To fix any polynomial of degree $N + 2$ one has to find $N + 3$ values at distinct points. From the explicit form of the transfer matrix (2.4) in terms of the R-matrix (2.1), the K-matrices (2.3) and (2.2), it is a simple exercise to find that

$$(6.4) \quad t(1) = (q + q^{-1})(\nu^+ + \nu^-)(\epsilon^+ + \epsilon^-) \prod_{i=1}^N b(qv_i) b(qv_i^{-1}),$$

$$(6.5) \quad t(i) = (q + q^{-1})(\nu^- - \nu^+)(\epsilon^- - \epsilon^+) \prod_{j=1}^N b(iqv_j) b(iqv_j^{-1}),$$

$$(6.6) \quad \lim_{u \rightarrow \infty} t(u) = \frac{u^{4+2N} q^{2+N}}{(q - q^{-1})^{2N}} (\tilde{\kappa}^2 \tilde{\tau}^2 + \kappa^2 \tau^2) + O(u^{2+2N})$$

which fixes 3 points. To get the remaining N points the properties of the R-matrix and the K-matrices, see [10] for the proof, allows to relate the transfer matrix with the Sklyanin determinant from the relation

$$(6.7) \quad t(q^{-1}v_i) t(v_i) = \frac{\text{Det}_q(K^+(q^{-1}v_i) K(q^{-1}v_i))}{b(qv_i^2) b(qv_i^{-2})}$$

where

$$(6.8) \quad \text{Det}_q(K^+(u) K(u)) = b(u^2) b(q^{-4}u^{-2}) \tilde{k}^+(qu) \tilde{k}^-(qu) \tilde{k}^+(q^{-1}u^{-1}) \tilde{k}^-(q^{-1}u^{-1}) \Lambda(qu) \Lambda(q^{-1}u^{-1}).$$

This relation is the core of the SoV and the ODBA methods. Using the polynomial properties of the transfer matrix, the $t(q^{-1}v_i)$ can be related to the $t(v_i)$ by Lagrange interpolation (see below the SoV characterization of the spectrum). It leads to N quadratic relations on the $t(v_i)$ and gives at most 2^N solutions from the Bézout theorem. The ODBA proposes an analytical ansatz for the Baxter T-Q relation that satisfy the conditions (6.1), (6.4), (6.5), (6.6) and (6.7) evaluated on some unknown eigenstate of the transfer matrix.

6.1. Connection with the ODBA. The key step in the ODBA was to propose a modified (or inhomogeneous) Baxter T-Q equation of the form

$$(6.9) \quad \Lambda^N(u, \bar{u}) Q^N(u, \bar{u}) = \psi(u) Q^N(q^{-1}u, \bar{u}) + \psi(q^{-1}u^{-1}) Q^N(qu, \bar{u}) + \chi c(u) c(q^{-1}u^{-1}) \Lambda(u) \Lambda(q^{-1}u^{-1})$$

with

$$(6.10) \quad Q^N(u) = G(u, \bar{u})^{-1}, \quad \psi(u) = \phi(u) \tilde{k}^+(u) \tilde{k}^-(u) \Lambda(u)$$

and the Bethe equation (4.12) ensuring the analyticity of the eigenvalues. The most important point is the addition of an inhomogenous term in the ansatz such that the conditions (6.1), (6.4), (6.5), (6.6) and (6.7) evaluated on some unknown eigenstate of the transfer matrix are satisfied. In the MABA context, the

relation (6.9) appears naturally as a consequence of the algebraic properties of the double-row monodromy operators, see (4.12).

In [10] another ansatz depending on the parity of the length of the chain N was proposed, actually it appears that there is an infinite family of modified Baxter T-Q equations which parametrize the spectrum of the transfer matrix. Here, from the MABA we recover the minimal solution with N Bethe roots (6.9) that is equivalent of (4.12). This minimal modified Baxter T-Q equation and the associated Bethe equations were shown from the SoV to give the complete description of the spectrum for the inhomogeneous case [20] and numerical evidence show that it should remains the case in the homogeneous limit.

Moreover, from the modified Baxter T-Q equation (6.9) and projection of the Bethe vector with the left SoV basis similar to the one we use here, the on-shell Bethe vector has been obtained in [33, 34]. The MABA gives the algebraic construction of the off-shell Bethe vector and allows, by limit or by imposing constraints between left and right boundary parameters, to recover the previous results in the literature, see [7] for details.

6.2. Connection with the SoV characterization of the spectrum. The SoV representation of the spectral problem of the transfer matrix is given [16] (up to some restrictions for the values of the boundary parameters and of the inhomogeneous parameters, see in [16]) by

$$(6.11) \quad \Lambda^N(u) = \sum_{j=1}^N \Lambda^N(v_j) g_j(u) + f(u)$$

with

$$(6.12) \quad g_j(u) = \frac{U(u)^2 - 1}{U(v_j)^2 - 1} \prod_{k=1, k \neq j}^N \frac{U(u) - U(v_k)}{U(v_j) - U(v_k)}$$

and

$$(6.13) \quad \begin{aligned} f(u) = & \Lambda^N(1) \frac{U(u) + 1}{U(1) + 1} \prod_{k=1}^N \frac{U(u) - U(v_k)}{1 - U(v_k)} + \Lambda^N(i) \frac{U(u) - 1}{U(i) - 1} \prod_{k=1}^N \frac{U(v_k) - U(u)}{1 + U(v_k)} \\ & + \frac{(q + q^{-1})^{N+2}}{(q - q^{-1})^{2N}} (\tilde{\kappa}^2 \tilde{\tau}^2 + \kappa^2 \tau^2) (U(u)^2 - 1) \prod_{k=1}^N (U(u) - U(v_k)), \end{aligned}$$

provided that the $\Lambda^N(v_j)$ satisfy the quadratic relations induced by the relation (6.7) and the interpolation formula (6.11) at $u = q^{-1}v_i$. It is proven in [20] that this representation is complete and that it is equivalent to the modified T-Q Baxter equation (6.9).

6.3. Connection with the SoV states. To give the SoV states, let us recall the right-SoV-basis used to construct the eigenstates of the transfer matrix. Following [16] we introduce the right pseudo-vacuum

$$(6.14) \quad |\Omega_m^N\rangle_{dl} = \otimes_{n=1}^N |Y(v_n, m+n)\rangle_{dl}$$

that is a pseudo-eigenstate of the \mathcal{B}_{dl} operator

$$(6.15) \quad \mathcal{B}_{dl}(u, m) |\Omega_m^N\rangle_{dl} = \tilde{\eta}_m^N \tilde{\Lambda}_b(u) |\Omega_{m+2}^N\rangle_{dl}$$

with

$$(6.16) \quad \tilde{\eta}_m^N = -q \tilde{\tau}^2 \left(q^{-N+m} \beta_{dl} - i \frac{\tilde{\mu}\tau}{\mu\tilde{\tau}} \right) \left(q^{-N+m} \beta_{dl} - i \frac{\mu\tau}{\tilde{\mu}\tilde{\tau}} \right) \frac{1}{\gamma_{m+1}^{dl}}$$

and

$$(6.17) \quad \tilde{\Lambda}_b(u) = u c(u) \prod_{n=1}^N b(quv_n) b(u/v_n).$$

The other states are constructed from the action of the operator

$$(6.18) \quad \hat{\mathcal{D}}(u, m) = \frac{\gamma_{m+1}}{\gamma_m} \langle \tilde{X}(u, m+2) | K(u) | Y(u^{-1}, m) \rangle$$

and are given by

$$(6.19) \quad |\tilde{\Psi}_m(\bar{h})\rangle_{dl} = \prod_{i=1}^N \left(\hat{\mathcal{D}}_{dl}(v_i, m) \right)^{1-h_i} |\Omega_m^N\rangle_{dl}.$$

The action of the $\mathcal{B}_{dl}(u, m)$ operator on these states is

$$(6.20) \quad \mathcal{B}_{dl}(u, m) |\tilde{\Psi}_m(\bar{h})\rangle_{dl} = \tilde{\eta}_m^N \tilde{\Lambda}_b(u) \prod_{i=1}^N \left(f(v_i, u) \right)^{1-h_i} |\tilde{\Psi}_{m+2}(\bar{h})\rangle_{dl}.$$

The SoV states are given in [16], up to an overall normalization, by

$$(6.21) \quad |\Phi\rangle_{SoV} = \sum_{\bar{h}} \frac{1}{\mu_m^N(\bar{h})} \prod_{j=1}^N \left(\frac{\Lambda^N(v_j^{-1})}{v_j \tilde{k}^+(v_j^{-1}) \phi(v_j^{-1})} \right)^{h_j} |\tilde{\Psi}_{m+2}(\bar{h})\rangle_{dl},$$

where the $\Lambda^N(v_j^{-1})$ are fixed from the quadratic relation (6.7) and the interpolation formula (6.11). We note that in [16] the gauge parameter β is arbitrary and we restrict here to the case $\beta = \beta_{dl}$ to make the link with the Bethe vector (4.2) with $M = N$. The $\mu_m^N(\bar{h})$ is the diagonal element of the scalar product between left and right SoV basis

$$(6.22) \quad {}_{dl} \langle \tilde{\Psi}_m(\bar{h}) | \tilde{\Psi}_{m+2}(\bar{k}) \rangle_{dl} = \mu_m^N(\bar{h}) \prod_{i=1}^N \delta_{h_i, k_i}$$

explicitly given by

$$(6.23) \quad \begin{aligned} \mu_m^N(\bar{h}) &= \prod_{i=1}^N \frac{\bar{\eta}_{m-2(i-1)}^N}{\gamma_{m+1+i}^{dl}} v_i^{-2h_i+1} c(v_i) b(q^{-2h_i+1} v_i^2) \\ &\times \prod_{j < i}^N b(q^{-2h_j+1} v_j/v_i) b(q^{-h_j-h_i+1} v_i/v_j) b(q^{-2h_j+1} v_i v_j) b(q^{h_j-h_i} v_i v_j) \end{aligned}$$

where

$$(6.24) \quad \bar{\eta}_m^N = \eta_m^N \frac{\gamma_{m+N}^{dl} \gamma_{m+1+N}^{dl}}{\gamma_m^{dl}}.$$

To relate this result with the on-shell Bethe vector (4.2) with $M = N$ constructed from the MABA we project both vectors with the left SoV-basis. For the Bethe vector we use the formula (5.8) with $u = u_i$ and for the SoV vector we use the scalar product between left and right SoV-basis (6.22). Thus we obtain in both cases, up to a overall constant, the same function, for each of the 2^N components, that show the equivalence of the two characterizations for the eigenstates of the transfer matrix.

7. CONCLUSION

In this paper we have proved the expression of the off-shell action of the modified creation operator on the Bethe vector, see proposition 4.1. Thus we give the proof of the off-shell action of the transfer matrix of the XXZ spin chain on the segment conjectured in [7] and before in some limit cases [5, 4]. The proof uses the left SoV basis [16] and exhibits similarities with the proof of the on-shell Bethe vector from the ODBA method [34].

This shows the deep relation between the MABA, the ODBA and the SoV methods. Indeed, the MABA explains the algebraic origin of the inhomogenous term in the modified T-Q Baxter equation introduced in

[10] from the ODBA, as a modification of the analytical Bethe ansatz. Moreover it provides a correspondence between the Bethe vector and the eigenvectors in the SoV framework, completing the link already done for the eigenvalues in [20]. Let us add that it implies the existence of a modified coordinate Bethe ansatz whose wave function can be obtained by projection of the on-shell Bethe vector on the appropriate basis, extending the constrained boundary cases obtained in [13, 14] to generic boundaries. Thus the MABA incorporates the different BA (ODBA, Analytical BA, ABA, Coordinate BA) and is equivalent in the inhomogeneous case to the SoV.

The SoV method yields a proof of the completeness of the spectrum, at least in the inhomogeneous case, which can be extended to the MABA and, by complementarity, the MABA provides a regularization for the SoV such that the homogeneous limit is well defined. We remark that such a connection was already used in the case of the XXX spin chain on the circle in [23], where the completeness of the Bethe ansatz solution was proved. Thus it seems that the complete description of the spectral problem of finite dimensional quantum integrable models constructed from the Yang-Baxter and reflection equations can be handled by both the Bethe ansatz and SoV methods.

It could be also of interest to relate the MABA for the XXZ spin chain on the segment with other known methods such as the q-Onsager approach [2] or the non-polynomial solution from the homogeneous Baxter T-Q relation [21].

The knowledge of the Bethe vector is the first step to consider the correlation functions of the XXZ spin chain on the segment in the ABA framework. An important question will be to understand the structure of the scalar product of the Bethe vector and see if some determinant representation can be obtained by analogy with the diagonal case [19]. Let us mention the recent algebraic functional approach [18] used in the diagonal case that should allow to find integral representations of the scalar product from our off-shell results.

Other interesting questions will be to apply the MABA to other models such as periodic spin chains without $U(1)$ symmetry where the ODBA [9, 33] and the SoV [31, 27] methods were applied as well as developing a Nested MABA for spin chains without $U(1)$ related to higher rank algebras where only the spectrum was obtained in few cases from the ODBA, see *e. g.* [11].

Acknowledgements: S.B. and R.A.P. thank R. Nepomechie for discussions. S.B. is supported by the Université de Cergy-Pontoise post doctoral fellowship. S.B. is also partially supported by São Paulo Research Foundation (FAPESP), grant # 2014/09832-1 and thanks the Departamento de Física of the Universidade Federal de São Carlos for hospitality where a part of this work was done. R.A.P. is supported by FAPESP, grants # 2014/00453-8 and # 2014/20364-0. N.G. is supported by the Université de Cergy-Pontoise.

APPENDIX A. FUNCTIONS

We use the following functions throughout the text,

$$(A.1) \quad b(u) = \frac{u - u^{-1}}{q - q^{-1}}, \quad k^-(u) = \nu_- u + \nu_+ u^{-1}, \quad k^+(u) = \epsilon_+ u + \epsilon_- u^{-1}, \quad c(u) = u^2 - u^{-2},$$

$$(A.2) \quad \phi(u) = \frac{b(q^2 u^2)}{b(qu^2)}, \quad G(u, v) = \frac{1}{b(u/v)b(quv)}, \quad \tilde{F}(u, v) = (v/u)G(u, v) \frac{b(q^2 u^2)}{\phi(v)},$$

$$(A.3) \quad f(u, v) = \frac{b(qv/u)b(uv)}{b(v/u)b(quv)}, \quad g(u, v) = \frac{\phi(q^{-1}v^{-1})}{b(u/v)}, \quad w(u, v) = -\frac{1}{b(quv)},$$

$$(A.4) \quad h(u, v) = \frac{b(q^2 uv)b(qu/v)}{b(quv)b(u/v)}, \quad k(u, v) = \frac{\phi(u)}{b(v/u)}, \quad n(u, v) = \frac{\phi(u)\phi(q^{-1}v^{-1})}{b(quv)},$$

$$(A.5) \quad \gamma(u, m) = \alpha q^{-m} u - \beta q^m u^{-1}, \quad \gamma(1, m) = \gamma_m,$$

$$(A.6) \quad g(u, v, m) = \frac{\gamma(u/v, m+1)}{\gamma_{m+1}} g(u, v), \quad w(u, v, m) = \frac{\gamma(uv, m)}{\gamma_{m+1}} w(u, v),$$

$$(A.7) \quad k(u, v, m) = \frac{\gamma(v/u, m+1)}{\gamma_{m+1}} k(u, v), \quad n(u, v, m) = \frac{\gamma(1/(uv), m+2)}{\gamma_{m+1}} n(u, v),$$

The following functional relations hold

$$(A.8) \quad f(u, \bar{u}) + \sum_{i=1}^M g(u, u_i, m-2) f(u_i, \bar{u}_i) - \phi(u_i) w(u, u_i, m-2) h(u_i, \bar{u}_i) = \frac{\gamma_{m-2M-1}}{\gamma_{m-1}},$$

and

$$(A.9) \quad h(u, \bar{u}) + \phi(u)^{-1} \sum_{i=1}^M \phi(u_i) k(u, u_i, m-2) h(u_i, \bar{u}_i) - n(u, u_i, m-2) f(u_i, \bar{u}_i) = \frac{\gamma_{m-2M-1}}{\gamma_{m-1}}.$$

To prove these functional relations we consider them as a function of u . One can easily check that the left hand side of these equations are holomorphic functions on the full complex plane, thus they are constant. To evaluate these constants one sends u to infinity and use the relation

$$(A.10) \quad \sum_{i=1}^M \phi(u_i) h(u_i, \bar{u}_i) + \phi(q^{-1} u_i^{-1}) f(u_i, \bar{u}_i) = \frac{q^{2M} - q^{-2M}}{q - q^{-1}}.$$

This relation can be proved in a similar way. The left hand side is a symmetric function of the set \bar{u} , so one can consider it as a single variable function of any u_k and show that for generic parameter \bar{u}_k this function is holomorphic on the full complex plane, thus it is constant. Then sending u_k to infinity one find a recursion relation on M that fix the constant.

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